

Bulk viscosity of superfluid neutron stars

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(Dated:)

The hydrodynamics, describing dynamical effects in superfluid neutron stars, essentially differs from the standard one-fluid hydrodynamics. In particular, we have four bulk viscosity coefficients in the theory instead of one. In this paper we calculate these coefficients, for the first time, assuming they are due to non-equilibrium beta-processes (such as modified or direct Urca process). The results of our analysis are used to estimate characteristic damping times of sound waves in superfluid neutron stars. It is demonstrated that all four bulk viscosity coefficients lead to comparable dissipation of sound waves and should be considered on the same footing.

PACS numbers: 97.60.Jd, 26.60.+c, 47.37.+q, 47.75.+f

I. INTRODUCTION

The matter in pulsating neutron stars is not (even locally) in chemical equilibrium. Particles of different kinds turn into one another so that the system evolves to equilibrium. If a deviation from the equilibrium is small then the processes of mutual transformations of particles can be described in terms of an effective bulk viscosity (see, e.g., Ref. [1]). This viscosity influences the ‘instability windows’, that are the regions of physical parameters (e.g., the rotation period and temperature of a star) at which the neutron star becomes unstable against the radiation of gravitational waves [2, 3, 4, 5]. The bulk viscosity, generated by non-equilibrium processes of particle transformations, was calculated in a series of papers for neutron-star matter of various composition (for example, for matter composed of neutrons, protons, and electrons with an admixture of muons; for hyperon or quark matter). A short review and references to these papers can be found in Ref. [6].

It is generally agreed that the stellar matter becomes superfluid at a certain stage of neutron star thermal evolution [7, 8, 9]. A lot of attention has been paid to the question how superfluidity affects the bulk viscosity (see, e.g., Refs. [10, 11, 12, 13, 14, 15]). In analogy with the ordinary hydrodynamics of non-superfluid liquid, the only one ‘standard’ bulk viscosity coefficient has been calculated and analyzed in all these papers. Meanwhile, it is well known [1, 16, 17] that a superfluid liquid, composed of identical particles, is generally described by the *four* bulk viscosity coefficients. So, what can be expected from neutron stars, which contain a mixture of many superfluid species?

In this paper we show that non-equilibrium processes of particle transformations lead to the appearance of at least four bulk viscosity coefficients. Each of them is important for analyzing dissipative processes in superfluid neutron stars. To be specific, we consider the simplest model of stellar matter composed of neutrons, protons, and electrons (npe-matter). In this case the bulk viscosity is associated with the non-equilibrium direct or modified Urca process.

The paper is organized as follows. In Sec. II we phe-

nomenologically obtain the general form of the dissipative corrections to the equations of relativistic hydrodynamics [18, 19], describing a superfluid liquid composed of identical particles. In Sec. III this dissipative hydrodynamics is generalized to describe superfluid mixtures and applied to npe-matter. In Sec. IV we calculate and analyse all four bulk viscosity coefficients provided by non-equilibrium beta-processes. For illustration of these results, in Sec. V we calculate the characteristic damping times of sound waves in superfluid npe-matter. Sec. VI presents summary.

II. THE DISSIPATIVE RELATIVISTIC HYDRODYNAMICS OF ONE-COMPONENT SUPERFLUID LIQUID

In this section we obtain the general form of dissipative terms entering the equations of relativistic superfluid hydrodynamics of electrically neutral liquid composed of identical particles. For that purpose, we need to choose a version of non-dissipative hydrodynamics. There is a number of equivalent formulations of non-dissipative relativistic superfluid hydrodynamics [18, 20, 21, 22, 23, 24]. The most elegant and general amongst them seems to be the formulation of Carter [20, 21, 22] in which the hydrodynamic equations follow from a convective variational principle. Most of the relativistic calculations (see Refs. [25, 26, 27]) modelling pulsations of superfluid neutron stars have been made within this approach (see, however, [19]).

In this paper, we will not use the Carter’s hydrodynamics because it is an essentially phenomenological theory and it does not allow easy interpretation in terms of quantities calculated from microscopic theory. Since our main goal is the calculation of bulk viscosity coefficients, we will employ the hydrodynamics of Son [18]. It was initially proposed in the context of heavy-ion collisions and is derived directly from microscopic theory. Therefore, it is straightforward to relate various parameters of this hydrodynamics to microphysics. Using the notations of Ref. [19] it can be rewritten in a particularly simple form, which is a natural relativistic generalization of the

standard non-relativistic superfluid hydrodynamics pioneered by Tisza [28], Landau [29, 30], and Khalatnikov [31].

Although the Son's description is ideal for comparing with microphysics, it has some serious disadvantages. In contrast to the Carter's hydrodynamics, in which the basic fluid variables are the particle number density current and the entropy density current, the Son's hydrodynamics is a hybrid in a sense that its fluid variables are the rescaled entropy density current and the rescaled momentum of a particle (or a Cooper pair) from the condensate (in the literature it is traditionally and somewhat confusedly referred to as 'the superfluid velocity'). As a consequence, the Landau-type hydrodynamics of Son has a *lower symmetry* than that of Carter. However, it can be simply translated into the hydrodynamics of Carter as was demonstrated, for example, in the review paper by Andersson and Comer [32] (see their section 16.2).

Below we will use the hydrodynamics of Son [18], employing the notations of Ref. [19], convenient for our problem.

Unless otherwise stated, the speed of light is set equal to $c = 1$.

The hydrodynamic equations take the standard form

$$\partial_\mu j^\mu = 0, \quad (1)$$

$$\partial_\mu T^{\mu\nu} = 0, \quad (2)$$

where $\partial^\mu \equiv \partial/\partial x_\mu$ (space-time indices are denoted by Greek letters). Neglecting dissipation, the particle current density j^μ and the energy-momentum tensor $T^{\mu\nu}$ can be presented in the form [19]

$$j^\mu = n u^\mu + Y w^\mu, \quad (3)$$

$$T^{\mu\nu} = (P + \varepsilon) u^\mu u^\nu + P \eta^{\mu\nu} + Y (w^\mu w^\nu + \mu w^\mu u^\nu + \mu w^\nu u^\mu). \quad (4)$$

Here n , P , ε , and μ (not to be confused with the space-time index μ !) are the number density, pressure, energy density, and relativistic chemical potential of particles, respectively; $\eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ is the special-relativistic metric; Y is the relativistic analogue of superfluid density ρ_s . In the non-relativistic limit, we have $Y = \rho_s/m^2$, where m is the mass of a free particle. Further, u^μ is the four-velocity of normal (non-superfluid) liquid component, normalized so that

$$u_\mu u^\mu = -1; \quad (5)$$

w^μ is the four-velocity, which describes the motion of superfluid liquid component. It can be expressed through some scalar function ϕ ,

$$w^\mu = \partial^\mu \phi - \mu u^\mu. \quad (6)$$

It is easy to verify that ϕ is related to the wave function phase Φ of the Cooper-pair condensate by the equality $\nabla\phi = (\hbar/2) \nabla\Phi$ (see Ref. [19]). In the non-relativistic

limit spatial components of the four-vectors u^μ and w^μ are equal to

$$\mathbf{u} = \mathbf{V}_q, \quad \mathbf{w} = m(\mathbf{V}_s - \mathbf{V}_q), \quad (7)$$

where $\mathbf{V}_s = \nabla\phi/m$ and \mathbf{V}_q are, respectively, the superfluid and normal velocities of the well known non-relativistic theory of superfluid liquids (see, e.g., Ref. [17]).

The four-velocity w^μ must satisfy one additional constraint. This constraint fixes a *comoving* frame, that is the frame where we measure (and define) all the thermodynamic quantities. Choosing the constraint in the form

$$u_\mu w^\mu = 0, \quad (8)$$

from Eqs. (3) and (4) one obtains that in this particular case *comoving* is the frame where four-velocity equals $u^\mu = (1, 0, 0, 0)$. It is straightforward to show that in this frame the basic thermodynamic quantities n , $\nabla\phi$, and ε are defined by

$$j^0 = n, \quad (9)$$

$$\mathbf{j} = Y \nabla\phi, \quad (10)$$

$$T^{00} = \varepsilon. \quad (11)$$

Using some equation of state and taking into account the second law of thermodynamics (T is the temperature, S is the entropy density)

$$d\varepsilon = T dS + \mu dn + \frac{Y}{2} d(w^\mu w_\mu), \quad (12)$$

as well as the definition of the pressure

$$P \equiv -\varepsilon + \mu n + TS, \quad (13)$$

we can express all other thermodynamic quantities as functions of ε , n , and $\nabla\phi$. Eqs. (1)–(13) fully describe the non-dissipative relativistic hydrodynamics of uncharged superfluid liquid. As a consequence of these formulae, one can easily derive the continuity equation for the entropy,

$$\partial_\mu (S u^\mu) = 0. \quad (14)$$

Now let us include dissipation in the hydrodynamics described above. As in the non-dissipative case, we assume that in the comoving frame [in which $u^\mu = (1, 0, 0, 0)$], the basic thermodynamic quantities ε , n , and $\nabla\phi$ are still defined by Eqs. (9)–(11). Being written in a relativistically invariant form, the conditions (9) and (10) imply that the particle current density j^μ is still given by Eq. (3), where some four-velocity w^μ satisfies the constraint (8) [as in the non-dissipative case]. In view of Eqs. (9) and (10), the most general expression for the four-velocity w^μ is

$$w^\mu \equiv \partial^\mu \phi - (\mu + \varkappa) u^\mu. \quad (15)$$

Here a scalar \varkappa is a small dissipative correction to be determined below. If we neglect dissipation, then $\varkappa = 0$ and Eq. (15) coincides naturally with (6).

Without any loss of generality, the expression for the energy-momentum tensor $T^{\mu\nu}$ can be presented in the form:

$$T^{\mu\nu} = (P + \varepsilon) u^\mu u^\nu + P \eta^{\mu\nu} + Y (w^\mu w^\nu + \mu w^\mu u^\nu + \mu w^\nu u^\mu) + \tau^{\mu\nu}. \quad (16)$$

Here $\tau^{\mu\nu}$ is an unknown dissipative tensor. In view of Eqs. (8) and (11), $\tau^{\mu\nu}$ satisfies the constraint

$$u_\mu u_\nu \tau^{\mu\nu} = 0. \quad (17)$$

Let us determine the dissipative corrections $\tau^{\mu\nu}$ and \varkappa assuming they are linear in small gradients of hydrodynamic variables. For this aim we need to derive an entropy generation equation. It can be easily obtained from Eqs. (1)–(2) if we make use of Eqs. (3), (8), (12), (13), (15), and (16),

$$\begin{aligned} \partial_\mu S^\mu = & -\frac{\varkappa}{T} \partial_\mu (Y w^\mu) - \tau^{\mu\nu} \partial_\mu \left(\frac{u_\nu}{T} \right) \\ & + Y w^\mu \frac{\varkappa}{T^2} \partial_\mu T + u^\nu Y w^\mu \frac{\varkappa}{T} \partial_\nu u_\mu. \end{aligned} \quad (18)$$

Here the entropy current density S^μ is given by

$$S^\mu = S u^\mu - \frac{u_\nu}{T} \tau^{\mu\nu} - \frac{\varkappa}{T} Y w^\mu, \quad (19)$$

and satisfies the natural constraint $u_\mu S^\mu = -S$.

Let us analyze the last two terms in Eq. (18). In addition to a quadratic dependence on small gradients of hydrodynamic variables, they also depend on the four-velocity w^μ . As follows from Eq. (7), the spatial part of the four-vector w^μ is proportional to the difference between the superfluid and normal velocities. In a large variety of problems this difference is small (as a consequence, the time component w^0 is also small because of the constraint 8). In particular, it cannot exceed some (not very large) critical value $\Delta \mathbf{V}_{\text{cr}}$ at which superfluidity breaks down (see Sec. IV). For instance, if we study small perturbations of matter which is initially at rest (in thermodynamic equilibrium with $u^\mu = 0$ and $w^\mu = 0$), then the last two terms in Eq. (18) are much smaller than the first two terms (a typical example of such situation is provided by sound waves in superfluid npe-matter, see Sec. V). Below, we will neglect the last two terms in Eq. (18) when obtaining the dissipative corrections $\tau^{\mu\nu}$ and \varkappa .

Moreover, in the dissipative corrections we will also neglect small dissipative terms, explicitly depending on w^μ . For example, we will neglect the terms of the form $w^\mu \partial_\mu T$ and $u^\mu w^\nu \partial_\gamma u^\gamma$ in the expressions for \varkappa and $\tau^{\mu\nu}$, respectively. An inclusion of these small terms would result in 13 kinetic coefficients describing dissipation in superfluid liquid. In the non-relativistic case the same approximation is used, for instance, in the textbook by Landau and Lifshitz [1] (see their §140) and in the monograph by Khalatnikov [17].

Since the entropy does not decrease, the right-hand side of Eq. (18) must be positive. This requirement puts certain restrictions on a general form of $\tau^{\mu\nu}$ and \varkappa (linear in gradients). The standard consideration (see, e.g., Ref. [33]) shows that

$$\begin{aligned} \tau^{\mu\nu} = & -\kappa (H^{\mu\gamma} u^\nu + H^{\nu\gamma} u^\mu) (\partial_\gamma T + T u^\delta \partial_\delta u_\gamma) \\ & - \eta H^{\mu\gamma} H^{\nu\delta} \left(\partial_\delta u_\gamma + \partial_\gamma u_\delta - \frac{2}{3} \eta_{\gamma\delta} \partial_\varepsilon u^\varepsilon \right) \\ & - \xi_1 H^{\mu\nu} \partial_\gamma (Y w^\gamma) - \xi_2 H^{\mu\nu} \partial_\gamma u^\gamma, \\ \varkappa = & -\xi_3 \partial_\mu (Y w^\mu) - \xi_4 \partial_\mu u^\mu. \end{aligned} \quad (20)$$

In these equations $H^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu$; κ and η are, respectively, the thermal conductivity and shear viscosity coefficients; ξ_1, \dots, ξ_4 are the bulk viscosity coefficients. From the Onsager symmetry principle it follows that

$$\xi_1 = \xi_4. \quad (22)$$

For the positive definiteness of the quadratic form in the right-hand side of Eq. (18) it is necessary to have the kinetic coefficients κ , η , ξ_2 , and ξ_3 positive and the coefficient ξ_1 satisfying the inequality

$$\xi_1^2 \leq \xi_2 \xi_3. \quad (23)$$

In the non-relativistic limit the dissipative hydrodynamics proposed here coincides with the well known theory of Khalatnikov (see, e.g., Refs. [1, 17, 34]). For illustration, let us indicate how the bulk viscosity coefficients $\xi_{\text{Kh}1}, \dots, \xi_{\text{Kh}4}$ of Khalatnikov are related to those introduced in this paper. It is easy to demonstrate that

$$\xi_{\text{Kh}1} = \frac{\xi_1}{m}, \quad \xi_{\text{Kh}2} = \xi_2, \quad (24)$$

$$\xi_{\text{Kh}3} = \frac{\xi_3}{m^2}, \quad \xi_{\text{Kh}4} = \frac{\xi_4}{m}. \quad (25)$$

Thus, in this section we have constructed the relativistic dissipative hydrodynamics of a superfluid liquid, composed of identical particles. It should be noted, that the dissipation was first included into the hydrodynamics [18] by Pujol and Davesne [35]. However, it is difficult to use their dissipative hydrodynamics in applications. The point is that the authors do not specify the *comoving* frame, where they define thermodynamic quantities. It is easy to verify that the frame which is defined as *comoving* in our paper, cannot serve as *comoving* in Ref. [35].

III. VISCOSITY IN SUPERFLUID MIXTURES

Let us apply the general formulae obtained in Sec. II to neutron star matter. As already mentioned in Sec. I, we consider the simplest model of neutron star cores composed of neutrons (n), protons (p), and electrons (e). All results of this and following sections can be easily generalized to the case of matter with more complicated

composition (e.g., npe-matter with admixture of muons or hyperon matter).

It is generally agreed that as a neutron star cools down the neutrons and protons become superfluid in its core. In such a system we have three velocity fields (instead of two, as in the previous section). They are superfluid velocity of neutrons, superfluid velocity of protons, and normal velocity u^μ of neutron and proton Bogoliubov quasiparticles and electrons. In this section we do not consider the dissipative effects related to the diffusion of particles. Neglecting the diffusion, nucleon Bogoliubov excitations and electrons move with the same velocity u^μ . In analogy with Sec. II, it is convenient to introduce four-velocities $w_{(n)}^\mu$ and $w_{(p)}^\mu$ instead of superfluid velocities of neutrons and protons, respectively. The Son's version of non-dissipative hydrodynamics was extended to the case of npe-mixture in Ref. [19] (for earlier formulations see, e.g., Refs. [25, 36, 37, 38, 39, 40]). The main goal of this section is to include the viscous dissipative terms into the hydrodynamics [19]. Below the subscripts i and k refer to nucleons, $i, k = n, p$. Unless otherwise stated, the summation is assumed over repeated nucleon indices i and k .

The full set of hydrodynamic equations describing superfluid mixtures consists of i) energy-momentum conservation law (2) with the energy-momentum tensor $T^{\mu\nu}$ given by

$$T^{\mu\nu} = (P + \varepsilon) u^\mu u^\nu + P \eta^{\mu\nu} + Y_{ik} \left[w_{(i)}^\mu w_{(k)}^\nu + \mu_i w_{(k)}^\mu u^\nu + \mu_k w_{(i)}^\nu u^\mu \right] + \tau^{\mu\nu}; \quad (26)$$

ii) particle conservation laws written for neutrons, protons, and electrons ($l = n, p, e$),

$$\partial_\mu j_{(l)}^\mu = 0, \quad j_{(i)}^\mu = n_i u^\mu + Y_{ik} w_{(k)}^\mu, \quad j_{(e)}^\mu = n_e u^\mu; \quad (27)$$

iii) constraints on the four-velocities $w_{(i)}^\mu$,

$$u_\mu w_{(i)}^\mu = 0, \quad (28)$$

and iv) the second law of thermodynamics,

$$d\varepsilon = T dS + \mu_i dn_i + \mu_e dn_e + \frac{Y_{ik}}{2} d \left[w_{(i)}^\mu w_{(k)\mu} \right]. \quad (29)$$

To take into account potentiality of superfluid motion, four-velocities $w_{(i)}^\mu$ should be expressed through some scalar functions ϕ_i and presented in the form

$$w_{(i)}^\mu = \partial^\mu \phi_i - q_i A^\mu - (\mu_i + \varkappa_i) u^\mu. \quad (30)$$

Note that one can avoid introduction of these new functions ϕ_i in the hydrodynamics of superfluid mixtures if one formulates the potentiality condition (30) in the equivalent way

$$\begin{aligned} & \partial^\nu \left[w_{(i)}^\mu + q_i A^\mu + (\mu_i + \varkappa_i) u^\mu \right] \\ &= \partial^\mu \left[w_{(i)}^\nu + q_i A^\nu + (\mu_i + \varkappa_i) u^\nu \right]. \end{aligned} \quad (31)$$

Below we will use the latter formulation because it is more suitable for our purpose. In this approach four-velocities $w_{(i)}^\mu$ should be treated as independent hydrodynamic variables.

In Eqs. (26)–(31) μ_l and n_l are, respectively, the relativistic chemical potential and the number density of particle species $l = n, p, e$; A^μ is the four-potential of the electromagnetic field; q_i is the electric charge of nucleon species i . Furthermore, $Y_{ik} = Y_{ki}$ is a 2×2 symmetric matrix which naturally appears in the theory as a generalization of the superfluid density to the case of superfluid mixtures. In the non-relativistic limit this matrix is related to the entrainment matrix ρ_{ik} (see Refs. [19, 41, 42, 43]) by the equality $Y_{ik} = \rho_{ik}/(m_i m_k)$, where m_i is the mass of nucleon species i . The pressure P is defined in the same way as for non-superfluid npe-matter (compare with Eq. 13),

$$P \equiv -\varepsilon + \mu_i n_i + \mu_e n_e + TS. \quad (32)$$

The dissipative hydrodynamics formulated above differs from the hydrodynamics [19], describing superfluid mixtures, only by the dissipative terms $\tau^{\mu\nu}$, \varkappa_n , and \varkappa_p . The general form of these terms can be found from the entropy generation equation which is analogous to Eq. (18) [see Sec. II],

$$\begin{aligned} \partial_\mu S^\mu &= -\frac{\varkappa_i}{T} \partial_\mu \left[Y_{ik} w_{(k)}^\mu \right] - \tau^{\mu\nu} \partial_\mu \left(\frac{u_\nu}{T} \right) \\ &+ Y_{ik} w_{(k)}^\mu \frac{\varkappa_i}{T^2} \partial_\mu T + u^\nu Y_{ik} w_{(k)}^\mu \frac{\varkappa_i}{T} \partial_\nu u_\mu, \end{aligned} \quad (33)$$

where the entropy density current S^μ is

$$S^\mu = S u^\mu - \frac{u_\nu}{T} \tau^{\mu\nu} - \frac{\varkappa_i}{T} Y_{ik} w_{(k)}^\mu. \quad (34)$$

Using the requirement that the entropy does not decrease, one can easily obtain the dissipative terms $\tau^{\mu\nu}$, \varkappa_n , and \varkappa_p from Eq. (33),

$$\begin{aligned} \tau^{\mu\nu} &= -\kappa (H^{\mu\gamma} u^\nu + H^{\nu\gamma} u^\mu) (\partial_\gamma T + T u^\delta \partial_\delta u_\gamma) \\ &- \eta H^{\mu\gamma} H^{\nu\delta} \left(\partial_\delta u_\gamma + \partial_\gamma u_\delta - \frac{2}{3} \eta_{\gamma\delta} \partial_\varepsilon u^\varepsilon \right) \\ &- \xi_{1i} H^{\mu\nu} \partial_\gamma \left[Y_{ik} w_{(k)}^\gamma \right] - \xi_2 H^{\mu\nu} \partial_\gamma u^\gamma, \end{aligned} \quad (35)$$

$$\varkappa_n = -\xi_{3i} \partial_\mu \left[Y_{ik} w_{(k)}^\mu \right] - \xi_{4n} \partial_\mu u^\mu, \quad (36)$$

$$\varkappa_p = -\xi_{5i} \partial_\mu \left[Y_{ik} w_{(k)}^\mu \right] - \xi_{4p} \partial_\mu u^\mu. \quad (37)$$

Here, as in Sec. II, we omit small dissipative terms, explicitly depending on $w_{(i)}^\mu$ and, in addition, we neglect particle diffusion. An inclusion of these terms would result in 19 kinetic coefficients as it has been recently demonstrated by Andersson and Comer [40] for the case of npe-matter (they obtained their result using a non-relativistic version of the Carter's hydrodynamics).

In Eqs. (35)–(37) ξ_{1i} , ξ_2 , ξ_{3i} , ξ_{4i} , and ξ_{5i} are the bulk viscosity coefficients ($i = n, p$). Some of them are related by the Onsager symmetry principle,

$$\xi_{1i} = \xi_{4i}, \quad \xi_{3p} = \xi_{5n}. \quad (38)$$

In addition, for positive definiteness of the quadratic form in the right-hand side of Eq. (33) one needs the following inequalities

$$\begin{aligned} \kappa &\geq 0, \quad \eta \geq 0, \quad \xi_{3n} \geq 0, \quad \xi_{5p} \geq 0, \quad \xi_2 \geq 0, \\ \xi_{5p}\xi_2 &\geq \xi_{1p}^2, \quad \xi_{3n}\xi_2 \geq \xi_{1n}^2, \quad \xi_{3n}\xi_{5p} \geq \xi_{3p}^2, \\ 2\xi_{1n}\xi_{1p}\xi_{3p} + \xi_2\xi_{3n}\xi_{5p} - \xi_{1p}^2\xi_{3n} - \xi_{3p}^2\xi_2 - \xi_{1n}^2\xi_{5p} &\geq 0. \end{aligned} \quad (39)$$

Equations (33)–(39) are derived under the assumption that electrons and protons can move independently. However, this is not the case since they are charged. Any macroscopic motion of electrons is accompanied by that of protons to ensure quasineutrality condition (see, e.g., [19]),

$$n_e = n_p. \quad (40)$$

One can obtain then from the continuity equations (27) for protons and electrons (neglecting small ‘diffusive’ terms),

$$\partial_\mu [Y_{pk} w_{(k)}^\mu] = 0. \quad (41)$$

In principle (if we are not interested in the distribution of the electromagnetic field, which couples together electrons and protons), we can use this equation instead of the constraints (28) and (31) for protons.

Equations (33)–(39) should be modified to take into account the conditions (40) and (41). As follows from Eq. (41), the only bulk viscosity coefficients which contribute to $\tau^{\mu\nu}$ and \varkappa_n (see Eqs. 35 and 36) are ξ_{1n} , ξ_2 , ξ_{3n} , and ξ_{4n} . Since we neglect the last two terms in the right-hand side of the entropy generation equation (33), only these four coefficients are responsible for dissipation of mechanical energy of macroscopic motion in superfluid npe-matter.

IV. NON-EQUILIBRIUM BETA-PROCESSES AND THE CALCULATION OF BULK VISCOSITY

In Sec. III we phenomenologically considered npe-matter and found the viscous terms in the relativistic hydrodynamic equations for superfluid mixtures. When doing this, we have ignored the fact that because of non-equilibrium beta-processes, the number of particles in the system is not conserved. In this section we demonstrate that the effect of non-equilibrium beta-processes is equivalent to the appearance of four effective bulk viscosity coefficients ξ_{1n} , ξ_2 , ξ_{3n} , and ξ_{4n} in the hydrodynamic equations.

In the npe-matter we have two types of beta-processes responsible for beta-equilibration. They are the direct Urca process and the modified Urca process. The powerful direct Urca process is open only for some equations of state with large symmetry energy (and at large enough densities, when $p_{Fn} \leq p_{Fp} + p_{Fe}$, p_{Fn} , p_{Fp} , and p_{Fe} being

the Fermi momenta of neutrons, protons, and electrons, respectively). The direct and inverse reactions of this process have the form (see, e.g., Refs. [7, 9, 44])

$$n \rightarrow p + e + \bar{\nu}, \quad p + e \rightarrow n + \nu. \quad (42)$$

Here ν and $\bar{\nu}$ stand for neutrino and antineutrino, respectively. If the direct Urca process is forbidden by momentum conservation, then the main mechanism of beta-equilibration is the modified Urca process [7, 9, 44]

$$N + n \rightarrow N + p + e + \bar{\nu}, \quad N + p + e \rightarrow N + n + \nu. \quad (43)$$

An additional nucleon $N = n, p$ here is needed to take an excess of momentum away and open the process.

The full thermodynamic equilibrium includes beta-equilibrium and we have [44]

$$\delta\mu \equiv \mu_n - \mu_p - \mu_e = 0. \quad (44)$$

In this case the number of direct reactions in a matter element per unit time is equal to the number of inverse reactions. Thus, the total number of particles of any species remains constant. In particular, the electron generation rate $\Delta\Gamma$ (that is the net number of electrons generated in beta-reactions in a unit volume per unit time) is zero, $\Delta\Gamma = 0$.

If we perturb the system, the condition (44) will not necessarily hold ($\delta\mu \neq 0$) and the direct and inverse reactions will not precisely compensate each other ($\Delta\Gamma \neq 0$). In this paper we assume that the deviation from the equilibrium is small, $\delta\mu \ll k_B T$. Then $\Delta\Gamma$ can be presented in the form [10, 11, 12, 45, 46]

$$\Delta\Gamma = \lambda \delta\mu. \quad (45)$$

Here λ is a function of various thermodynamic quantities defined in the equilibrium state (e.g., of the temperature and the particle number densities). For superfluid npe-matter this function was calculated by Haensel, Levenfish, and Yakovlev [10, 11]. Note, that these authors neglected the dependence of $\Delta\Gamma$ on the scalars $w_{(i)}^\alpha w_{(k)\alpha}$ though (in principle) they can be non-zero in thermodynamic equilibrium. However, it seems that the results of Refs. [10, 11] are accurate as long as (as an example, we take the case of $i = k = n$)

$$\frac{w_{\mu(n)} w_{(n)}^\mu}{m_n} \sim m_n (\mathbf{V}_{sn} - \mathbf{V}_q)^2 \ll k_B T, \quad (46)$$

where \mathbf{V}_{sn} is the superfluid velocity of neutrons. A numerical estimate of Eq. (46) gives

$$|\Delta\mathbf{V}_n| \equiv |\mathbf{V}_{sn} - \mathbf{V}_q| \ll 0.01 c \left(\frac{T}{10^9 \text{K}} \right)^{1/2}. \quad (47)$$

For clarity, we introduce the velocity of light c in this condition. On the other hand, as follows from the Landau

criterion (see, e.g., Ref. [47]), superfluidity of neutrons breaks down if $|\Delta\mathbf{V}_n| > |\Delta\mathbf{V}_{cr}|$, where

$$|\Delta\mathbf{V}_{cr}| \sim \frac{\Delta_n}{p_{Fn}} \sim \frac{k_B T_{cn}}{p_{Fn}} \approx 0.0003 c \left(\frac{T_{cn}}{10^9 \text{K}} \right) \left(\frac{n_0}{n_n} \right)^{1/3}. \quad (48)$$

Here Δ_n is the energy gap in the neutron dispersion relation; T_{cn} is the critical temperature of neutron superfluidity onset; $n_0 = 0.16 \text{ fm}^{-3}$ is the number density of nucleons in saturated nuclear matter. It is worth noting that the criterion (48) is actually an *upper* limit on $|\Delta\mathbf{V}_{cr}|$. In reality, a superfluid state can be destroyed at much lower $|\Delta\mathbf{V}_{cr}|$ due to the formation of vortices in superfluid matter [16, 48]. Comparing Eqs. (47) and (48) one can see that if the neutrons are superfluid then at not very low temperatures the condition (47) is always justified.

To calculate the bulk viscosity coefficients we will use the *non-dissipative* hydrodynamics of superfluid mixtures [19] (see also Sec. III). The energy-momentum tensor $T^{\mu\nu}$ for such a hydrodynamics is given by Eq. (26), while the four-velocity $w_{(i)}^\mu$ satisfies the conditions (28) and (31). Notice, that the dissipative components $\tau^{\mu\nu}$ and \varkappa_i in Eqs. (26) and (31) should be taken zero, $\tau^{\mu\nu} = 0$ and $\varkappa_i = 0$. Further, we assume that the quasineutrality condition (40) is fulfilled in both equilibrated and non-equilibrated matter. Using Eq. (40), the second law of thermodynamics for mixtures (29) can be rewritten in the form:

$$d\varepsilon = T dS + \mu_n dn_b - \delta\mu dn_e + \frac{Y_{ik}}{2} d[w_{(i)}^\mu w_{(k)\mu}]. \quad (49)$$

Finally, let us assume that the four-velocities $w_{(i)}^\mu = 0$ in equilibrium.

To take the non-equilibrium beta-processes into consideration it is necessary to add corresponding sources in the right-hand sides of the continuity equations for electrons, protons, and neutrons,

$$\partial_\mu (n_e u^\mu) = \Delta\Gamma, \quad (50)$$

$$\partial_\mu [n_p u^\mu + Y_{pk} w_{(k)}^\mu] = \Delta\Gamma, \quad (51)$$

$$\partial_\mu [n_n u^\mu + Y_{nk} w_{(k)}^\mu] = -\Delta\Gamma. \quad (52)$$

When writing Eqs. (50)–(52) we bear in mind that every neutron decay is accompanied by the appearance of an electron and a proton (see the reactions 42 and 43).

Taking into account the quasineutrality condition (40), one gets from Eqs. (50) and (51) the equality (41). Thus, Eq. (41) remains the same as in the absence of beta-processes. It is more convenient to use the continuity equation for baryons instead of Eqs. (51) and (52), because non-equilibrium beta-processes do not influence the total number of baryons per unit volume, $n_b = n_n + n_p$. Summing together Eqs. (51) and (52) and using Eq. (41), one obtains

$$\partial_\mu [n_b u^\mu + Y_{nk} w_{(k)}^\mu] = 0. \quad (53)$$

Let us assume that npe-matter is slightly perturbed out of thermodynamic equilibrium so that deviations from the equilibrium are small and one can linearize the hydrodynamic equations. Below we will work in the comoving (at one particular moment) frame associated with some element of npe-matter. In such a frame $u^\mu = (1, 0, 0, 0)$. We can assume further that perturbations in the comoving frame depend on time t as $\exp(i\omega_c t)$, where ω_c is the frequency of perturbation (measured in this frame).

From the normalization condition (5) and Eq. (28) it follows that

$$w_{(i)}^0 = 0, \quad (54)$$

$$\partial_t u^0 = 0, \quad \partial_t w_{(i)}^0 = 0. \quad (55)$$

Using these equalities, one gets from Eqs. (50) and (53)

$$\partial_t n_e + \text{div}(n_e \mathbf{u}) = \Delta\Gamma, \quad (56)$$

$$\partial_t n_b + \text{div}[n_b \mathbf{u} + Y_{nk} \mathbf{w}_{(k)}] = 0. \quad (57)$$

Here \mathbf{u} and $\mathbf{w}_{(k)}$ are the spatial components of four-vectors u^μ and $w_{(k)}^\mu$, respectively. The number densities of electrons n_e and baryons n_b can be presented as $n_e = n_{e0} + \delta n_e$, $n_b = n_{b0} + \delta n_b$, where n_{e0} and n_{b0} are the equilibrium number densities, while δn_e and δn_b are small non-equilibrium terms depending on time as $\exp(i\omega_c t)$. Here and hereafter the thermodynamic quantities related to the equilibrium state will be denoted by the subscript ‘0’. Using these notations as well as formula (45) and linearizing Eqs. (56)–(57), we get

$$\delta n_e = \frac{1}{i\omega_c} [\lambda \delta\mu - n_{e0} \text{div}(\mathbf{u})], \quad (58)$$

$$\delta n_b = -\frac{1}{i\omega_c} \{n_{b0} \text{div}(\mathbf{u}) + \text{div}[Y_{nk} \mathbf{w}_{(k)}]\}. \quad (59)$$

Notice, that the chemical potential disbalance $\delta\mu$ in Eq. (58) depends on δn_e and δn_b . Actually, $\delta\mu$ can generally be presented as a function of n_b , n_e , T , and the scalars $w_{\mu(i)} w_{(k)}^\mu$ (the proton number density is equal to the electron one, see the quasineutrality condition 40). One can neglect the temperature dependence of $\delta\mu$ in a strongly degenerate npe-matter (see, e.g., Refs. [49, 50]). Moreover, since the scalars $w_{\mu(i)} w_{(k)}^\mu$ are of the second order smallness, their contribution to $\delta\mu$ is also negligible (we recall that $w_{(i)}^\mu = 0$ in equilibrium). Expanding $\delta\mu(n_b, n_e)$ in Taylor series in the vicinity of its equilibrium value (which is zero, $\delta\mu(n_{b0}, n_{e0}) = 0$, see Eq. 44), one obtains in the first approximation

$$\delta\mu(n_b, n_e) = \frac{\partial\delta\mu(n_{b0}, n_{e0})}{\partial n_{b0}} \delta n_b + \frac{\partial\delta\mu(n_{b0}, n_{e0})}{\partial n_{e0}} \delta n_e. \quad (60)$$

Analogous formulae can be written for perturbations of pressure $\delta P \equiv P(n_b, n_e) - P_0$, neutron chemical potential $\delta\mu_n \equiv \mu_n(n_b, n_e) - \mu_{n0}$, and energy density $\delta\varepsilon \equiv$

$$\varepsilon(n_b, n_e) - \varepsilon_0,$$

$$\delta P = \frac{\partial P(n_{b0}, n_{e0})}{\partial n_{b0}} \delta n_b + \frac{\partial P(n_{b0}, n_{e0})}{\partial n_{e0}} \delta n_e, \quad (61)$$

$$\delta \mu_n = \frac{\partial \mu_n(n_{b0}, n_{e0})}{\partial n_{b0}} \delta n_b + \frac{\partial \mu_n(n_{b0}, n_{e0})}{\partial n_{e0}} \delta n_e, \quad (62)$$

$$\delta \varepsilon = \frac{\partial \varepsilon(n_{b0}, n_{e0})}{\partial n_{b0}} \delta n_b. \quad (63)$$

In the last equation we have neglected the term of the form $[\partial \varepsilon(n_{b0}, n_{e0})/\partial n_{e0}] \delta n_e$. From the second law of thermodynamics (49) we have $\partial \varepsilon(n_{b0}, n_{e0})/\partial n_{e0} = -\delta \mu$. Therefore, this term is quadratically small and can be omitted.

Using Eqs. (58)–(60) one finds

$$\delta n_e = \frac{1}{F} \left\{ i n_{e0} \omega_c \operatorname{div}(\mathbf{u}) + \frac{\partial \delta \mu(n_{b0}, n_{e0})}{\partial n_{b0}} n_{b0} \lambda \operatorname{div}(\mathbf{u}) + \frac{\partial \delta \mu(n_{b0}, n_{e0})}{\partial n_{b0}} \lambda \operatorname{div} [Y_{nk} \mathbf{w}_{(k)}] \right\}, \quad (64)$$

where $F \equiv \omega_c^2 + i \omega_c \lambda \partial \delta \mu(n_{b0}, n_{e0})/\partial n_{e0}$. For non-equilibrium Urca-processes and typical (for neutron stars) pulsation frequencies $\omega_c \sim 10^3 - 10^4 \text{ s}^{-1}$, we have (see, e.g., Refs. [10, 11]) $\omega_c \gg \lambda |\partial \delta \mu(n_{b0}, n_{e0})/\partial n_{e0}|$. In this case Eq. (64) can be simplified by keeping only terms linear in λ . The result can be written as

$$\delta n_e = \delta n_{e1} + \delta n_{e2}, \quad (65)$$

where the first term equals

$$\delta n_{e1} = \frac{i n_{e0}}{\omega_c} \operatorname{div}(\mathbf{u}) \quad (66)$$

and describes compression and decompression of the pulsating matter. This term remains the same even in the absence of non-equilibrium beta-processes. The second term δn_{e2} is due to non-equilibrium beta-processes

$$\delta n_{e2} = \frac{\lambda}{\omega_c^2} \left\{ n_{b0} \frac{\partial \delta \mu(n_{b0}, x_{e0})}{\partial n_{b0}} \operatorname{div}(\mathbf{u}) + \frac{\partial \delta \mu(n_{b0}, n_{e0})}{\partial n_{b0}} \operatorname{div} [Y_{nk} \mathbf{w}_{(k)}] \right\}. \quad (67)$$

Notice, that in this formula the partial derivative $\partial \delta \mu(n_{b0}, x_{e0})/\partial n_{b0}$ is taken at constant value of $x_{e0} \equiv n_{e0}/n_{b0}$. When obtaining Eq. (67) we used the identity

$$n_{b0} \frac{\partial \Psi(n_{b0}, x_{e0})}{\partial n_{b0}} = n_{b0} \frac{\partial \Psi(n_{b0}, n_{e0})}{\partial n_{b0}} + n_{e0} \frac{\partial \Psi(n_{b0}, n_{e0})}{\partial n_{e0}}, \quad (68)$$

where Ψ is an arbitrary function of n_{b0} and n_{e0} .

Our further strategy is as follows. We take the energy-momentum tensor $T^{\mu\nu}$ for superfluid mixtures (with $\tau^{\mu\nu}=0$) from Eq. (26) and expand all the thermodynamic quantities (e.g., the pressure P and the energy density ε), which determine this tensor, around their equilibrium values. Restricting ourselves to linear perturbation

terms, we obtain for the tensor $T^{\mu\nu}$ (in the comoving frame),

$$\begin{aligned} T^{00} &= \varepsilon_0 + \delta \varepsilon, \\ T^{0j} &= T^{j0} = \mu_{i0} Y_{ik} w_{(k)}^j, \\ T^{jm} &= (P_0 + \delta P) \delta_{jm}. \end{aligned} \quad (69)$$

Here, the spatial indices j and m are equal to 1, 2, 3; the relativistic entrainment matrix Y_{ik} is taken in equilibrium; δP and $\delta \varepsilon$ are given by Eqs. (61) and (63), respectively.

Let us assume for a while that there are no non-equilibrium beta-processes in the matter. In this case $\delta n_{e2} = 0$ (see Eq. 67) and the matter is reversibly pulsating around the equilibrium. Then the mechanical energy is not dissipating, and the entropy is conserved. Thus it is obvious that dissipative are only those terms in the tensor $T^{\mu\nu}$ which are directly related to δn_{e2} . Writing out these terms in the form of a separate tensor $\tau_{\text{bulk}}^{\mu\nu}$, we have

$$\begin{aligned} \tau_{\text{bulk}}^{00} &= 0, \\ \tau_{\text{bulk}}^{0j} &= \tau_{\text{bulk}}^{j0} = 0, \\ \tau_{\text{bulk}}^{jm} &= \frac{\partial P(n_{b0}, n_{e0})}{\partial n_{e0}} \delta n_{e2} \delta_{jm}. \end{aligned} \quad (70)$$

This tensor can be easily rewritten in an arbitrary frame if we take into account Eq. (67),

$$\begin{aligned} \tau_{\text{bulk}}^{\mu\nu} &= \frac{\lambda}{\omega_c^2} \frac{\partial P(n_{b0}, n_{e0})}{\partial n_{e0}} H^{\mu\nu} \\ &\times \left\{ \frac{\partial \delta \mu(n_{b0}, n_{e0})}{\partial n_{b0}} \partial_\gamma [Y_{nk} w_{(k)}^\gamma] + n_{b0} \frac{\partial \delta \mu(n_{b0}, x_{e0})}{\partial n_{b0}} \partial_\gamma u^\gamma \right\}. \end{aligned} \quad (71)$$

Comparing the tensor $\tau_{\text{bulk}}^{\mu\nu}$ with the phenomenological dissipative tensor $\tau^{\mu\nu}$ (see Eq. 35), we find the expressions for the effective bulk viscosity coefficients ξ_{1n} and ξ_2 , generated by non-equilibrium beta-processes

$$\xi_{1n} = -\frac{\lambda}{\omega_c^2} \frac{\partial P(n_{b0}, n_{e0})}{\partial n_{e0}} \frac{\partial \delta \mu(n_{b0}, n_{e0})}{\partial n_{b0}}, \quad (72)$$

$$\xi_2 = -\frac{\lambda}{\omega_c^2} n_{b0} \frac{\partial P(n_{b0}, n_{e0})}{\partial n_{e0}} \frac{\partial \delta \mu(n_{b0}, x_{e0})}{\partial n_{b0}}. \quad (73)$$

Let us do the same with the potentiality condition (31) on the four-velocity of neutrons $w_{(n)}^\mu$. As a result, we obtain in the comoving frame the dissipative component \varkappa_n , appearing because of non-equilibrium beta-processes,

$$\varkappa_n = \frac{\partial \mu_n(n_{b0}, n_{e0})}{\partial n_{e0}} \delta n_{e2}. \quad (74)$$

In a fully covariant form, this component is given by Eq. (36) where the effective bulk viscosity coefficients ξ_{3n} and

ξ_{4n} are

$$\xi_{3n} = -\frac{\lambda}{\omega_c^2} \frac{\partial \mu_n(n_{b0}, n_{e0})}{\partial n_{e0}} \frac{\partial \delta \mu(n_{b0}, n_{e0})}{\partial n_{b0}}, \quad (75)$$

$$\xi_{4n} = -\frac{\lambda}{\omega_c^2} n_{b0} \frac{\partial \mu_n(n_{b0}, n_{e0})}{\partial n_{e0}} \frac{\partial \delta \mu(n_{b0}, x_{e0})}{\partial n_{b0}}, \quad (76)$$

and, in addition, the condition (41) is taken into account.

Thus, we have calculated the four effective bulk viscosity coefficients ξ_{1n} , ξ_2 , ξ_{3n} , and ξ_{4n} . As will be shown in Sec. V, each of them makes a comparable contribution to characteristic damping times of mechanical energy. Notice, that only the coefficient ξ_2 is usually analyzed in the literature devoted to non-equilibrium beta-processes in superfluid matter. The expression (73) for ξ_2 coincides with earlier results (see, e.g., Refs. [10, 11]).

Not all of the coefficients (72)–(73) and (75)–(76) are independent. The coefficients ξ_{1n} and ξ_{4n} are equal because of the Onsager principle (38). This can be shown if one applies the following relation for npe-matter (see, e.g., Ref. [10]),

$$\frac{\partial P(n_{b0}, x_{e0})}{\partial x_{e0}} = -n_{b0}^2 \frac{\partial \delta \mu(n_{b0}, x_{e0})}{\partial n_{b0}}. \quad (77)$$

Furthermore, it is easy to verify, that instead of one of the inequalities (39) relating the coefficients ξ_{1n} , ξ_2 , and ξ_{3n} , we have the strict equality

$$\xi_{1n}^2 = \xi_2 \xi_{3n}. \quad (78)$$

It is fulfilled only for those non-equilibrium processes, for which the expansion (65) is valid. Therefore, we have only *two* independent bulk viscosity coefficients.

To prove Eq. (78) it is instructive to consider the entropy generation equation. Neglecting all the dissipative processes except for the non-equilibrium beta-processes (e.g., neglecting thermal conductivity, diffusion, shear viscosity), one can obtain from the hydrodynamics discussed in this section

$$T \partial_\mu S^\mu = \delta \mu \Delta \Gamma = \lambda \delta \mu^2. \quad (79)$$

Here we made use of Eq. (45). Since we are interested only in terms linear in λ , we can substitute δn_{e1} for δn_e into Eq. (60) which determines $\delta \mu$.

On the other hand, the entropy generation equation in terms of the effective bulk viscosities takes the form (see Eqs. 33, (35)–(37), and 41),

$$T \partial_\mu S^\mu = \xi_{3n} \left\{ \partial_\mu \left[Y_{nk} w_{(k)}^\mu \right] \right\}^2 + (\xi_{1n} + \xi_{4n}) \partial_\mu \left[Y_{nk} w_{(k)}^\mu \right] \partial_\mu u^\mu + \xi_2 (\partial_\mu u^\mu)^2. \quad (80)$$

Let us compare the right-hand sides of Eqs. (79) and (80). It follows from Eqs. (60) and (79) that for any given u^μ it is always possible to choose four-velocities $w_{(k)}^\mu$ in such a way, that $\delta \mu = 0$ and the entropy generation rate vanishes (at some point and at some particular moment).

In terms of the bulk viscosity formalism this means that one can vanish the quadratic form in the right-hand side of Eq. (80) by an appropriate choice of these velocities. This is possible *only* if the equality (78) is satisfied.

V. DAMPING OF SOUND WAVES IN SUPERFLUID NPE-MATTER

Let us illustrate the results of previous sections by calculating characteristic damping times of sound waves propagating in a homogeneous superfluid npe-matter. For simplicity, we consider only the damping due to the effective bulk viscosity. Neglecting dissipation, the sound modes of superfluid npe-matter have been thoroughly investigated starting from the pioneering paper by Epstein [51] in which he argued that there would be two types of sound modes in neutron stars (see, e.g., Refs. [19, 38, 39]). In particular, Gusakov and Andersson [19] were the first who considered in full relativity sound modes in npe-matter at finite temperatures. Here we closely follow their analysis. The pulsation equations (81) and (82) of Ref. [19] can be used to describe sound waves taking into account dissipation. Thus, there is no need to derive these equations from the hydrodynamics of superfluid mixtures (Secs. III and IV) once again. Instead, we will rewrite them using the notations adopted in our paper. The result is

$$\partial_t [(P_0 + \varepsilon_0) \mathbf{u} + \mu_{n0} Y_{nk} \mathbf{w}_{(k)}] = -\nabla \delta P, \quad (81)$$

$$\partial_t [\mu_{n0} \mathbf{u} + \mathbf{w}_{(n)}] = -\nabla \delta \mu_n. \quad (82)$$

The first equation is a consequence of the relativistic Euler equation, which can be derived from Eq. (2) in a standard way (see, e.g., Ref. [1]). The second equation follows from the condition (31) written for neutrons. To fully define the system, Eqs. (81) and (82) should be supplemented by the condition (41). Using Eqs. (54) and (55), this condition can be presented in the form

$$\text{div} [Y_{pk} \mathbf{w}_{(k)}] = 0. \quad (83)$$

Now assuming that all the perturbations are plane waves proportional to $\exp(i\omega t - i\mathbf{k}\mathbf{r})$, one obtains the following compatibility condition for Eqs. (81)–(83) [$s = \omega/k$ is the velocity of sound in units of c]

$$y s^4 + C_1 s^2 + C_2 + \delta A = 0, \quad (84)$$

where

$$y = \frac{Y_{pp} n_{b0}}{\mu_{n0} (Y_{nn} Y_{pp} - Y_{np} Y_{pn})} - 1, \quad (85)$$

$$C_1 = \left[\frac{P_0}{\mu_{n0} n_{b0}} (\beta_1 - \gamma_1 - \gamma_1 y) + \gamma_2 - \beta_2 \right], \quad (86)$$

$$C_2 = \frac{P_0}{\mu_{n0} n_{b0}} (\beta_2 \gamma_1 - \beta_1 \gamma_2), \quad (87)$$

$$\gamma_1 = \frac{n_{b0}}{P_0} \frac{\partial P(n_{b0}, x_{e0})}{\partial n_{b0}}, \quad \gamma_2 = \frac{n_{b0}}{\mu_{n0}} \frac{\partial \mu_n(n_{b0}, x_{e0})}{\partial n_{b0}}, \quad (88)$$

$$\beta_1 = \frac{n_{b0}}{P_0} \frac{\partial P(n_{b0}, n_{e0})}{\partial n_{b0}}, \quad \beta_2 = \frac{n_{b0}}{\mu_{n0}} \frac{\partial \mu_n(n_{b0}, n_{e0})}{\partial n_{b0}}. \quad (89)$$

A small complex term δA appears in the compatibility condition (84) because of the bulk viscosity. It is given by

$$\delta A = -\frac{i\omega}{\mu_{n0}^2 n_{b0}} (A_1 + s^2 A_2), \quad (90)$$

$$A_1 = \mu_{n0} n_{b0} \gamma_2 \xi_{1n} - \mu_{n0} \beta_2 \xi_2 - P_0 n_{b0} \gamma_1 \xi_{3n} + P_0 \beta_1 \xi_{4n}, \quad (91)$$

$$A_2 = \mu_{n0} (y \xi_2 + \xi_2 + n_{b0}^2 \xi_{3n} - n_{b0} \xi_{1n} - n_{b0} \xi_{4n}). \quad (92)$$

We remind that the bulk viscosity coefficients (and the quantities A_1 and A_2) depend on the frequency ω , $A_{1,2} \sim \omega^{-2}$. The biquadratic equation (84) has two non-trivial solutions for two possible sound velocities. Neglecting dissipation, these modes have been analyzed in details in Ref. [19]. In particular, the sound velocities $s_1^{(0)}$ and $s_2^{(0)}$ have been calculated there for the first and second modes. The dissipation leads to the appearance of small complex corrections $\delta s_{1,2}$ to the velocities $s_{1,2}^{(0)}$ and consequently to decrements of sound waves. Since δA is small in comparison with other terms in Eq. (84), one can use the perturbation theory in deriving the characteristic damping times $\tau_{1,2}$. The parameters τ_1 and τ_2 are e-folding times of the pulsation amplitude for the first and second sound modes, respectively,

$$\tau_{1,2} \approx \frac{i}{k \delta s_{1,2}} = -\frac{2i s_{1,2}^{(0)}}{k \delta A} \left(2y s_{1,2}^{(0)2} + C_1 \right). \quad (93)$$

As follows from Eqs. (90)–(92), they are *independent* of ω .

At $T \rightarrow T_{cn}$ we have $Y_{nn}, Y_{np}, Y_{pn} \rightarrow 0$ and $y \approx n_{b0}/(\mu_{n0} Y_{nn}) \rightarrow \infty$ (see Ref. [19] for a more detailed discussion). In this limit the characteristic damping times are

$$\tau_1 \approx \frac{2 P_0 \gamma_1}{\omega^2 \xi_2}, \quad (94)$$

$$\tau_2 \approx -\frac{2 \mu_{n0} P_0 \gamma_1 D}{\omega^2 (\gamma_1 A_1 + \mu_{n0} D \xi_2)}. \quad (95)$$

Here we introduce the parameter $D \equiv \beta_2 \gamma_1 - \beta_1 \gamma_2$. At $T > T_{cn}$ the neutrons are non-superfluid. In this case the second mode does not exist [formally, $s_2^{(0)} = 0$], while the first mode is the usual sound wave. The characteristic

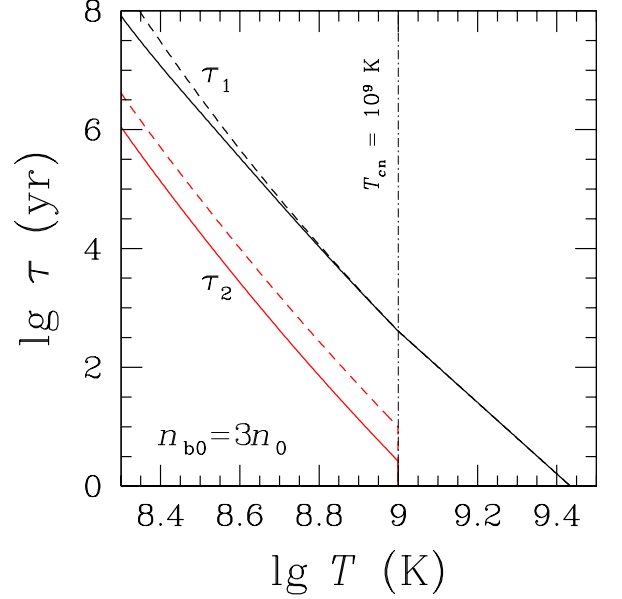


FIG. 1: Characteristic damping times $\tau_{1,2}$ of sound waves versus temperature T for the first mode (two upper curves) and for the second mode (two lower curves). The solid curves demonstrate the damping of sound taking into account all four bulk viscosity coefficients. The dashed curves are calculated assuming that only ξ_2 is non-zero. The neutron critical temperature is indicated by the vertical dot-dashed line. The baryon number density is $n_b = 3n_0 = 0.48 \text{ fm}^{-3}$.

damping time for an ordinary sound wave is given by Eq. (94). As expected, its damping is governed by the only one bulk viscosity coefficient ξ_2 .

For illustration, in Fig. 1 we present the characteristic damping times $\tau_{1,2}$ of sound waves (in years) as a function of temperature T for two sound modes. The figure is plotted for npe-matter with the baryon number density $n_{b0} = 3n_0$. The critical temperature of neutrons is taken to be $T_{cn} = 10^9 \text{ K}$. The protons are assumed to be non-superfluid. When calculating thermodynamic quantities and their derivatives we employed the equation of state from Ref. [52]. It opens the direct Urca process at baryon number density of $5.84n_0$. Therefore, the process is forbidden for $n_{b0} = 3n_0$. In this case, the main mechanism of energy dissipation is the non-equilibrium modified Urca process. To calculate the function λ , which enters Eqs. (72)–(73) and (75)–(76) for the bulk viscosity coefficients, we have used the results of Ref. [11]. For calculating the relativistic entrainment matrix Y_{ik} we have employed the BJ v6 nucleon-nucleon potential [19, 43, 53]. Actually, the microphysics input we use here to plot the figure is taken from Ref. [19] (see this reference for more details).

The two upper curves correspond to the first sound mode, while the two lower curves – to the second mode. The solid curves are plotted taking into account all four bulk viscosity coefficients. The dashed curves are ob-

tained under the assumption that all coefficients but ξ_2 are equal to zero, $\xi_{1n} = \xi_{3n} = \xi_{4n} \equiv 0$.

As seen from the figure, the characteristic damping times of sound waves increase as the temperature decreases. This is natural, because when the neutrons are superfluid, the Urca processes (and hence the bulk viscosity) are exponentially suppressed at $T \ll T_{cn}$. Let us emphasize that at temperatures $T \lesssim 5 \times 10^8$ K, the shear viscosity of electrons can exceed the bulk viscosity generated by the non-equilibrium modified Urca process. As a result, the damping of sound waves will be mainly due to the shear viscosity.

The first mode turns into the ordinary sound at $T > T_{cn}$. As follows from the figure, in the vicinity of neutron critical temperature the dissipation is primarily determined by the bulk viscosity coefficient ξ_2 (in accordance with Eq. 94). Consequently, near the transition point the solid and the dashed curves for the first mode coincide. On the contrary, the difference between the solid and the dashed curves for the second mode remains significant at any $T < T_{cn}$. The characteristic damping times for these two curves differ approximately by a factor of 3.

It is worth noting that we would come to the similar conclusions if we considered sound waves in denser matter, where the direct Urca process is open. In that case the characteristic damping times would be 6–7 orders of magnitude smaller, but the relative difference between the solid and the dashed curves will be approximately the same.

Therefore, the main result of the present section is that the bulk viscosity coefficients ξ_{1n} , ξ_{3n} , and ξ_{4n} essentially influence the dissipative properties of superfluid npe-matter and cannot be ignored. All four bulk viscosity coefficients should be considered on the same footing.

VI. SUMMARY

We performed a self-consistent analysis of the influence of non-equilibrium beta-processes on dissipation of mechanical energy in superfluid matter of neutron stars. We start with the Son's version of non-dissipative one-fluid relativistic hydrodynamics to describe superfluid mixtures (see Refs. [18, 19]). We determined the general form of dissipative terms entering the equations of this hydrodynamics. For simplicity, the effects of particle diffusion were ignored. The equations of dissipative hydrodynamics were applied to the matter composed of neutrons, protons, and electrons (npe-matter). In this

case the hydrodynamic equations contain *four* bulk viscosity coefficients rather than one, as in non-superfluid matter.

It was demonstrated, that non-equilibrium beta-processes generate all four bulk viscosity coefficients, and only two of them are independent. The other two coefficients can be expressed through the first two by Eqs. (38) and (78). It is worth to emphasize that only the bulk viscosity coefficient ξ_2 has been considered in the astrophysical literature so far. The expression (73) for ξ_2 coincides with similar expressions of previous works (see, e.g., Refs. [10, 11, 45, 46]).

To illustrate the results obtained in the present paper we considered a problem of damping of sound waves via the bulk viscosity due to non-equilibrium beta-processes in superfluid homogeneous npe-matter. It was shown that all four bulk viscosity coefficients make comparable contributions to the characteristic damping times of sound waves.

Our results can be important for the analysis of various gravitational-driven instabilities in neutron stars, in particular, the r-mode instability (see, e.g., Ref. [4]). The additional bulk viscosity coefficients lead to a more effective damping of these instabilities. Moreover, the results can be applied to the problems of rotochemical and gravitochemical heating of millisecond pulsars with superfluid cores. In the absence of superfluidity these problems were carefully analyzed in Refs. [49, 54, 55, 56]. The first attempt to discuss qualitatively the effects of superfluidity has been made in Ref. [57].

In conclusion let us note that the method of the bulk viscosity calculation, used here in the simple case of npe-matter, can be extended to matter with more complicated composition (npe-matter with admixture of muons, hyperon or quark matter). Such a generalization is beyond the scope of the present study and will be considered elsewhere.

Acknowledgments

The author is grateful to D.P. Barsukov and E.M. Kantor for discussions, to A.I. Chugunov for technical assistance, to D.G. Yakovlev for reading the manuscript and critical comments, and to A. Reisenegger and anonymous referee for very useful remarks. This research was supported by RFBR (grants 05-02-16245 and 05-02-22003) and by the Federal Agency for Science and Innovations (grant NSh 9879.2006.2).

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